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A GENERALIZED GOAL IN RESTRICTED SUBSET SELECTION THEORY

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May 1974

A GENERALIZED GOAL IN RESTRICTED SUBSET SELECTION THEORY

by

Thomas J. Santner

Prepared under Contract N00014-67-A-0226-0014, Office of Naval Research at Purdue University. Work performed by the author during the spring of 1973 at Purdue University and subsequent visit in 1974.

A Generalized Goal in Restricted Subset Selection Theory

by

Thomas J. Santner
Cornell University

SUMMARY

A class of multiple decision procedures which selects a random size subset of populations not exceeding m (determined a priori) in size has previously been considered for the problem of selecting the best population among k candidates. The present paper generalizes the earlier results to the goal of selecting at least one of the t best populations. Applications of the basic theory can be made to many specific problems considered in the literature. In the regular case these include selection from univariate normal populations for means and variances, from gamma populations for scale factors and from noncentral chi square or noncentral F populations for noncentrality parameters and in the nonregular case selection from uniform populations with different supports.

Some key words: Ranking and selection; Multiple decision procedure; Restricted subset selection; Normal means; Noncentral F populations.

1. Introduction and Summary

Let $(\mathcal{X},\mathcal{B},P_i)$, $i = 1,\ldots,k$ be k probability spaces which, as is usual in the literature, shall be referred to as populations and denoted by $\pi_1,\ldots,\pi_k$. For this paper $\mathcal{X}$ is a finite dimensional Euclidean space, $\mathcal{B}$ is its Borel sigma field and P_i is an unknown probability measure from $\mathcal{P}$ a specified parametric family of probability measures on $(\mathcal{X},\mathcal{B})$. The worth of each π_i is characterized by an unknown scalar $\lambda_i = (P_i) \varepsilon \Lambda$ a known interval on the real line. Let $\lambda_{[1]} \leq \cdots \leq \lambda_{[k]}$ be the ordered λ_i's, $\pi_{(i)}$ the (unknown) population with parameter $\lambda_{[i]}$ and $\Omega = \{\underset{\sim}{\lambda} = (\lambda_1,\ldots,\lambda_k) | \lambda_i \varepsilon \Lambda$ for all $i\}$ the space of all possible underlying configurations of λ_i's.

Historically the first approach to this problem was the indifference zone formulation of Bechhofer (1954) in which the experimenter selects a fixed number of populations and his goal is guaranteed to be met with at least probability P^* whenever the true $\underset{\sim}{\lambda}$ lies outside some subset or zone of indifference of the parameter space Ω. In contrast to the indifference zone approach Gupta (1956,65) proposed the subset selection formulation in which a random size (between one and k) subset of the k populations is selected such that there is at least probability P^* of achieving the experimenter's goal no matter what the actual $\underset{\sim}{\lambda} \varepsilon \Omega$. The specific goal studied in this paper is selection of at least one of the t best populations $\pi_{(k-t+1)},\ldots,\pi_{(k)}$. Previous work on this same topic includes Mahamunulu (1967), Desu and Sobel (1968), Sobel (1969) and Panchapakesan (1969). It is assumed there is no a priori knowledge of the correct pairing of the π_i and $\pi_{(i)}$.

A class of single sample restricted subset selection procedures is proposed for this goal which gives more flexibility to the experimenter than does either the fixed subset size or random subset procedures by allowing him to specify, before experimentation, an upper bound m on the number of populations included in the selected subset. However should the data clearly indicate that a particular population is one of the t best, this rule retains the advantage of the subset selection procedure over the fixed size subset rule in allowing selection of fewer than m populations. Restricted subset selection procedures were introduced for the normal means problem in Gupta and Santner (1973). The notation and the rule $R(n)$ of the present paper follows Santner (1974) where a general theory for the goal $t = 1$ was given. Hence, as will be noted in Section 4, several properties of the proposed rule can be found in the earlier paper. Both subset selection $(t > k-m)$ and indifference zone $(t \leq k-m)$ probability requirements will be studied.

In Section 2 the problem will be formalized and a class of procedures proposed for its solution. In Section 3 the infimum of the probability of a correct selection is simultaneously derived for both the subset selection and indifference zone cases. The location or scale parameter problems are seen to yield a one stage minimization and sufficient conditions are given to allow its derivation in the general case. Section 4 gives properties of the proposed rule while Section 5 gives applications of the theory to selection from normal, noncentral F and uniform populations. Tables are included to allow implementation of the normal means procedure.

2. Formulation of the Problem

Each π_i yields iid observations $\{X_{ij}\}_{j=1}^{n}$ which are also independent between populations. It is assumed there exists a sequence of Borel measurable functions $\{T_n\}$ with $T_n: X^n \to \Lambda$ and such that $T_{in} = T_n(X_{i1},\ldots,X_{in})$ converges in probability to a monotone function (independent of i) of λ_i as $n \to \infty$. Also T_{in} has cdf $G_n(y|\lambda_i)$ with support $E_n^{\lambda_i}$ which is absolutely continuous with pdf $g_n(y|\lambda_i)$. For each n $\{G_n(y|\lambda)\,|\,\lambda \in \Lambda\}$ is assumed to be a stochastically increasing family. Let $p: \Lambda \to R$ and $h_n: \bigcup_{\lambda} E_n^{\lambda} \to R$ satisfy respectively (2.1) and (2.2) of Santner (1974), the main conditions of which are $p(\lambda) < \lambda$ for all $\lambda \in \Lambda$; $h_n(x) > -\infty$ for all n and $h_n(x) \to x$ as $n \to \infty$ for all x.

An indifference zone will be defined in Ω by means of the function $p(\lambda)$. Let

$$\Omega^t(p) = \{\underset{\sim}{\lambda} \in \Omega \,|\, \lambda_{[k-t]} \le p(\lambda_{[k-t+1]})\}$$

$$\Omega_0^t(p) = \{\underset{\sim}{\lambda} \,|\, \lambda_{[1]} = \lambda_{[k-t]} = p(\lambda_{[k-t+1]}), \lambda_{[k-t+1]} = \lambda_{[k]}\}$$

$$\Omega_0 = \{\underset{\sim}{\lambda} \,|\, \lambda_{[1]} = \lambda_{[k]}\} .$$

For each $\{h_n(x)\}$ define the rule

$$\underline{R(n)}: \text{ Select } \pi_i \Longleftrightarrow T_{in} \ge \max\{T_{[k-m+1]};\ h_n^{-1}(T_{[k]})\} \qquad (2.1)$$

where $T_{[1]} \le \cdots \le T_{[k]}$ are the ordered estimators and the n has been suppressed from $T_{[i]n}$ for ease of notation.

Example 2.1. For $p(\lambda) = \lambda - \delta$, $\delta > 0$, and $h_n(x) = x + d/\sqrt{n}$.

$$\Omega^t(p) = \{\underset{\sim}{\lambda} | \lambda_{[k-t+1]} - \lambda_{[k-t]} \geq \delta\}$$

$$\Omega_0^t(p) = \{\underset{\sim}{\lambda} | \lambda_{[1]} + \delta = \lambda_{[k-t]} + \delta = \lambda_{[k-t+1]} = \lambda_{[k]}\}$$

$$R(n): \text{ Select } \pi_i \iff T_{in} \geq \max\{T_{[k-m+1]}, T_{[k]} - d/\sqrt{n}\}$$

Goal I. Given P^*, $p(\lambda)$, t and $R(n)$ find the smallest common sample size n needed to satisfy the probability requirement

$$P_{\underset{\sim}{\lambda}}[CS|R(n)] \geq P^* \text{ for all } \underset{\sim}{\lambda} \in \Omega^t(p) . \tag{2.2}$$

The event $[CS|R(n)]$ occurs iff the selected subset contains at least one of $\pi_{(k-t+1)},\ldots,\pi_{(k)}$.

Similarly given P^*, n and $t > k-m$ a subset selection probability requirement (over all Ω) can be achieved by use of a class of rules $R(h)$ defined for $h \in H = \{h_d: \bigcup_\lambda E_n^\lambda \to R | d \in [0,\infty)\}$ where

(1) $h_d(x) > x$ for all x, d

(2) $h_0(x) \equiv x$

(3) For every x, $h_d(x)$ is continuous in d

(4) For every x, $h_d(x) \to x$ as $d \to \infty$.

For each $h = h_d \in H$, $R(h)$ is defined by (2.1) using h rather than h_n.

Goal S. Given P^*, n, t find $h \in H$ such that

$$P_{\underset{\sim}{\lambda}}[CS|R(h)] \geq P^* \text{ for all } \underset{\sim}{\lambda} \in \Omega . \tag{2.3}$$

Remark 2.1. Intuitively the subset selection approach is applicable since selection of m populations guarantees that a correct selection has occurre and hence (2.3) holds.

3. Probability of a Correct Selection

Since the form of the rules $R(n)$ and $R(h)$ are the same and a correct selection occurs for either iff at least one of the t best populations is selected the following result is applicable to both. Let $\mathcal{D} = \{k-t+1,\ldots,k\}$ and $\bar{\mathcal{D}} = \{1,\ldots,k-t\}$.

Theorem 3.1

$$P_{\underset{\sim}{\lambda}}[CS|R] \text{ is } \begin{cases} \uparrow \text{ in } \lambda_{[i]} \text{ for any } i \in \mathcal{D} \text{ when all other } \lambda_{[j]}\text{'s are fixed} \\ \downarrow \text{ in } \lambda_{[i]} \text{ for any } i \in \bar{\mathcal{D}} \text{ when all other } \lambda_{[j]}\text{'s are fixed.} \end{cases}$$

Proof. As usual let $P_{\underset{\sim}{\lambda}}[CS|R] = E_{\underset{\sim}{\lambda}}[n(\underset{\sim}{T})]$ where

$$n(\underset{\sim}{T}) = \begin{cases} 1, & T_{(j)} \geq \max\{T_{[k-m+1]}, h^{-1}(T_{[k]})\} \text{ for some } j \in \mathcal{D} \\ 0, & \text{otherwise .} \end{cases}$$

Then from a lemma of Mahamunulu (1967) and Alam and Rizvi (1966) it suffices to show $n(\underset{\sim}{T}) \uparrow$ in $T_{(i)}$ for any $i \in \mathcal{D}$ when all other $T_{(j)}$'s are fixed and $n(\underset{\sim}{T}) \downarrow$ in $T_{(i)}$ for any $i \in \bar{\mathcal{D}}$ when all other $T_{(j)}$'s are fixed. To prove the former pick any $\underset{\sim}{T}$ with $n(\underset{\sim}{T}) = 1$, fix $i \in \mathcal{D}$ and choose $\underset{\sim}{T}'$ having $T'_{(i)} > T_{(i)}$ and $T'_{(j)} = T_{(j)}$ for all $j \neq i$. $n(\underset{\sim}{T}) = 1$ implies

there exists $j_0 \in \mathcal{D}$ with $h(T_{(j_0)}) \geq T_{[k]}$ and $T_{(j_0)} \geq T_{[k-m+1]}$.

Subcase 1. $i = j_0$. It can be seen that $T'_{(j_0)} > T_{(j_0)}$ implies $T'_{(j_0)} \geq T'_{[k-m+1]}$ and $h(T'_{(j_0)}) \geq T'_{[k]}$ hence $\eta(\underset{\sim}{T}') = 1$.

Subcase 2. $i \neq j_0$. Again straightforward arguments show that either

(1) $T'_{(j_0)} \geq \max\{T'_{[k-m+1]}, h^{-1}(T'_{[k]})\}$

or

(2) $T'_{(i)} \geq \max\{T'_{[k-m+1]}, h^{-1}(T'_{[k]})\}$

hence $\eta(\underset{\sim}{T}') = 1$. The second part of the proof follows along the same lines.

Hence the infimum of the probability of a correct selection occurs when for all $i \in \overline{\mathcal{D}}$, $\lambda_{[i]}$ is as large as possible and when for all $i \in \mathcal{D}$, $\lambda_{[i]}$ is as small as possible.

Corollary 3.1. For $t \leq k-m$, $\inf\limits_{\Omega^t(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)] = \inf\limits_{\Omega_0^t(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)]$ and for $t > k-m$, $\inf\limits_{\Omega} P_{\underset{\sim}{\lambda}}[CS|R(h)] = \inf\limits_{\Omega_0} P_{\underset{\sim}{\lambda}}[CS|R(h)]$.

These last two probabilities can be expressed as follows. Let

$$I(y;\, a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^y w^{a-1}(1-w)^{b-1}dw$$

denote the incomplete beta function with parameters a and b. Also for all $\underset{\sim}{\lambda}^* \in \Omega_0^t(p)$ $\exists$ $\lambda \in \Lambda' = \{\lambda \in \Lambda | p(\lambda) \in \Lambda\}$ with $\underset{\sim}{\lambda}^* = (p(\lambda),\ldots,p(\lambda),\lambda,\ldots,\lambda)$ and then

$$P_{\underset{\sim}{\lambda}^*}[CS|R(n)] = P[Z \geq \max\{T_{[k-m+1]}, h^{-1}(T_{[k]})\}] \quad \text{where}$$

$$Z = \max\{T_{(k-t+1)},\dots,T_{(k)}\}$$

$$= \sum_{i=k-m+1}^{k} P[Z = T_{[i]}, h_n(T_{[i]}) \geq T_{[k]}] \quad \text{since}$$

$$t \leq k-m \Rightarrow [Z \geq T_{[k-m+1]}] = \bigcup_{i=k-m+1}^{k} [Z = T_{[i]}]$$

$$= \sum_{i=k-m+1}^{k} \sum_{j=k-t+1}^{k} P[Z = T_{(j)} = T_{[i]}, h_n(T_{(j)}) \geq T_{[k]}] .$$

But

$$[Z = T_{(j)} = T_{[i]}] = [T_{(j)} > T_{(\ell)} \text{ for } (i-t)\ \ell\text{'s with } \ell \in \bar{\mathcal{D}};\ T_{(j)} < T_{(\ell)}$$

$$\text{for } (k-i)\ \ell\text{'s with } \ell \in \bar{\mathcal{D}};\ T_{(j)} > T_{(\ell)}\ \forall\ \ell \in \mathcal{D}\backslash\{j\}]$$

$$= \bigcup_{\nu=1}^{\binom{k-t}{i-t}} [T_{(j)} > T_{(\ell)}\ \forall\ \ell \in S_\nu^i \cup \mathcal{D}\backslash\{j\};\ T_{(j)} < T_{(\ell)}\ \forall\ \ell \in \bar{S}_\nu^i],$$

where $\{S_\nu^i\}$ is the collection of all subsets of size $(i-t)$ from $\bar{\mathcal{D}}$ and $\bar{S}_\nu^i = \bar{\mathcal{D}}\backslash S_\nu^i$. So

$$P_{\underset{\sim}{\lambda}^*}[CS|R(n)] = \sum_{i=k-m+1}^{k} \sum_{j=k-t+1}^{k} \sum_{\nu=1}^{\binom{k-t}{i-t}} P\begin{bmatrix} T_{(\ell)} < T_{(j)}\ \forall\ \ell \in S_\nu^i \cup \mathcal{D}\backslash\{j\}; \\ \\ T_{(j)} < T_{(\ell)} < h_n(T_{(j)})\ \forall\ \ell \in \bar{S}_\nu^i \end{bmatrix}$$

$$= \sum_{i=k-m+1}^{k} t\binom{k-t}{i-t} \int_{E_n^\lambda} \{G_n(y|\lambda)\}^{t-1}\{G_n(h_n(y)|p(\lambda)) - G_n(y|p(\lambda))\}^{k-i} \cdot$$

$$\cdot \{G_n(y|p(\lambda))\}^{i-t} dG_n(y|\lambda) \qquad (3.1)$$

$$= \beta(\lambda,t,n) \quad \text{where}$$

$$\beta(\lambda,t,n) = \int_{E_n^\lambda} \{G_n(h_n(y)|p(\lambda))\}^{k-t} I\left\{\frac{G_n(y|p(\lambda))}{G_n(h_n(y)|p(\lambda))};\ k-m-t+1,m\right\} d\{G_n(y|\lambda)\}^t .$$

Hence for $t \leq k-m$

$$\inf_{\Omega^t(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)] = \inf_{\lambda\epsilon\Lambda'} \beta(\lambda,t,n) . \tag{3.2}$$

Similar arguments show for $t > k-m$

$$\inf_{\Omega} P_{\underset{\sim}{\lambda}}[CS|R(h)] = \inf_{\lambda\epsilon\Lambda} \gamma(\lambda,t,h) \tag{3.3}$$

where

$$\gamma(\lambda,t,h) = \int_{E_n^\lambda} \{G_n(h(y)|\lambda)\}^{k-t} d\{G_n(y|\lambda)\}^t .$$

Remark 3.1. If the selection problem is a location or scale parameter problem then a change of variables shows $\beta(\lambda,t,n)$ and $\gamma(\lambda,t,h)$ are independent of λ and hence (3.2) and (3.3) are completely evaluated. The normal means problem will provide an example.

When $\beta(\lambda,t,n)$ depends on λ and $G_n(y|\lambda)$ and $p(\lambda)$ satisfy regularity conditions (3.3) of Santner (1974) then the following conditions imply $\beta(\lambda,t,n)$ is monotone and gives the evaluation of (3.2). This result generalizes Theorem 3.2 of the earlier paper.

Theorem 3.2. If all derivatives in (3.5) and (3.6) exist and for all $\lambda \in \Lambda'$

$$E_n^\lambda = E_n \tag{3.4}$$

$$g_n(y|\lambda) \frac{\partial G_n(h_n(y)|p(\lambda))}{\partial\lambda} - h_n'(y) g_n(h_n(y)|p(\lambda)) \frac{\partial G_n(y|\lambda)}{\partial\lambda} \geq 0 \text{ ae} \tag{3.5}$$

$$g_n(y|\lambda) \frac{\partial G_n(y|p(\lambda))}{\partial\lambda} - g_n(y|p(\lambda)) \frac{\partial G_n(y|\lambda)}{\partial\lambda} \geq 0 \quad \text{ae} \tag{3.6}$$

then $\beta(\lambda,t,n)$ is nondecreasing in λ.

The proof of the result follows from an application of Theorem 2.1 of Gupta and Panchapakesan (1972) where, in their notation, we have taken

$$F_\lambda(y) = \{G_n(y|\lambda)\}^t$$

$$\phi(y,\lambda) = \{G_n(h_n(y)|p(\lambda))\}^{k-t} I\left\{\frac{G_n(y|p(\lambda))}{G_n(h_n(y)|p(\lambda))}; k-m-t+1, m\right\}.$$

Hence if there is a smallest $\lambda_0 \in \Lambda'$ and (3.4)-(3.6) hold then

$$\inf_{\Omega^t(p)} P[CS|R(n)] = \beta(\lambda_0,t,n).$$

Remark 3.2. If (3.5) and (3.6) are both nonpositive then $\beta(\lambda,t,n)$ is nonincreasing and (3.2) can again be evaluated.

Remark 3.3. If (3.4) and (3.6) hold for all $\lambda \in \Lambda$ with $h_n(y)$ replaced by $h(y)$ then $\gamma(\lambda,t,h)$ is nondecreasing and hence (3.3) can be computed. Panchapakesan (1969) gives this result for the case $m = k$.

4. Properties of the Rules

When $t \leq k-m$ we establish the consistency of any sequence of rules $\{R(n)\}$ in the sense that given any $(P^*,p(\lambda))$ requirement there exists a sample size n which attains (2.2).

Theorem 4.1. When $t \leq k-m$ if there exists $\lambda_0 \in \Lambda'$ and $N \geq 1$ such that

$$\inf_{\Lambda'} \beta(\lambda,t,n) = \beta(\lambda_0,t,n) \quad \text{for all } n \geq N \tag{4.1}$$

then $\inf_{\Omega^t(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)] \to 1$ as $n \to \infty$.

Proof. From Section 3 and (4.1) it suffices to show $\beta(\lambda_0,t,n) \to 1$ as $n \to \infty$. Since $p(\lambda_0) < \lambda_0$ $\exists$ $\alpha \ni p(\lambda_0) < \alpha < \lambda_0$. Let

$$f_{n1i}(y) = \{G_n(y|\lambda_0)\}^{t-1}$$

$$f_{n2i}(y) = \{G_n(y|p(\lambda_0))\}^{i-t}$$

$$f_{n3i}(y) = \{G_n(h_n(y)|p(\lambda_0)) - G_n(y|p(\lambda_0))\}^{k-i}$$

then by (3.1) if $\int \prod_{j=1}^{3} f_{nji}(y)dG_n(y|\lambda_0) \to 0$ for $i < k$ and $\to 1/t$ for $i = k$ as $n \to \infty$ the result holds.

Case A. $k-m+1 \le i \le k-1$. Given $0 < \varepsilon < 1$ $\exists$ $M \ge N \ni \begin{cases} G_n(\alpha|\lambda_0) < \varepsilon/2 \\ G_n(\alpha|p(\lambda_0)) > 1 - \varepsilon/2 \end{cases}$

for $n \ge M$ since $T_{in} \xrightarrow{p} \lambda_0$ as $n \to \infty$. So for all $y \ge \alpha$

$$\begin{aligned} \prod_{j=1}^{3} f_{nji}(y) &\le f_{n3i}(y) \\ &\le \{G_n(h_n(y)|p(\lambda_0)) - G_n(y|p(\lambda_0))\} \\ &\le 1 - G_n(\alpha|p(\lambda_0)) \\ &< \varepsilon/2 \text{ for all } n \ge M . \end{aligned}$$

Hence for all $n \ge M$,

$$0 \le \int_{-\infty}^{\infty} \prod_{j=1}^{3} f_{nji}(y)dG_n(y|\lambda_0)$$

$$\le \int_{-\infty}^{\alpha} 1dG_n(y|\lambda_0) + \int_{\alpha}^{\infty} \varepsilon/2dG_n(y|\lambda_0) < \varepsilon .$$

Case B. $i = k$. Since $f_{n3k}(y) \equiv 1 \Rightarrow t\int \prod_{j=1}^{3} f_{njk}(g)dG_n(y|\lambda_0) \le$ $t\int \{G_n(y|\lambda_0)\}^{t-1}dG_n(g|\lambda_0) = 1$. Given $0 < \varepsilon' < 1$ choose $\varepsilon \ni (1-\varepsilon)^{k-t}(1-\varepsilon^t) = 1 - \varepsilon'$ and $M \ge N \ni$ for all $n \ge M$, $G_n(\alpha|\lambda_0) < \varepsilon$ and $G_n(\alpha|p(\lambda_0)) > 1 - \varepsilon$.

So for all $n \ge M$,

$$1 \ge t\int \prod_{j=1}^{2} f_{njk}(y)dG_n(y|\lambda_0) \ge t\int_{\alpha}^{\infty} \prod_{j=1}^{2} f_{njk}(y)dG_n(y|\lambda_0)$$

$$\ge t\int_{\alpha}^{\infty} \{G_n(y|\lambda_0)\}^{t-1}(1-\varepsilon)^{k-t}dG_n(y|\lambda_0)$$

$$= (1-\varepsilon)^{k-t}[1 - G_n(\alpha|\lambda_0)^t]$$

$$\ge (1-\varepsilon)^{k-t}(1-\varepsilon^t) = 1-\varepsilon'$$ and completes the proof.

Remark 4.1. Similarly if there exists $\lambda_0 \in \Lambda$ such that $\inf_{\Lambda} \gamma(\lambda,t,h) = \gamma(\lambda_0,t,h)$ for all $h \in H$ then $\lim_{d\to\infty} P_\lambda[CS|R(h)] = 1$. Hence (2.3) can always be achieved by a rule in the class $\{R(h)|h \in H\}$.

As mentioned in Section 1 since monotonicity properties of $R(n)$ and other indicators of its performance such as the number of selected populations, $S(n)$; its expected value, $E[S(n)]$; $\sup_{\Omega} E[S(n)]$ and convergence properties of $\{S(n)\}$ do not depend on the goal, these results can be found in Santner (1974).

5. Applications

This section contains several examples of applications of the results of this paper to selection from specific populations.

I. Suppose $\pi_i \sim N(\mu_i,\sigma^2)$, $i = 1,\ldots,k$ where the common σ^2 is known and the experimenter is interested in selecting at least one of the t populations with largest μ_i's. Hence $\lambda_i = \mu_i$ and taking T_{in} $\bar{X}_i = \frac{1}{n}\sum_{j=1}^{n} X_{ij}$ gives $G_n(y|\mu_i) = \Phi(n^{\frac{1}{2}}(y-\mu_i)/\sigma)$ where Φ is the cdf of a $N(0,1)$ random variable. Since π_i is a location parameter family we take $p(\mu) = \mu - \delta$ and $h_n(x) = x + d\sigma/\sqrt{n}$

$$\Longrightarrow \begin{cases} \Omega^t(p) = \{\underset{\sim}{\mu} | \mu_{[k-t+1]} - \mu_{[k-t]} \geq \delta\} \\ R(n): \text{ Select } \pi_i \Longleftrightarrow \bar{X}_i \geq \max\{\bar{X}_{[k-m+1]}, \bar{X}_{[k]} - d\sigma/\sqrt{n}\} . \end{cases} \tag{5.1}$$

Applying (3.2) and a change of variables gives

$$\inf_{\Omega^t(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)] = \int_{-\infty}^{\infty} \{\Phi(y + d + \frac{\sqrt{n}\,\delta}{\sigma})\}^{k-t} \cdot I\left[\frac{\Phi(y + \frac{\sqrt{n}\,\delta}{\sigma})}{\Phi(y + d + \frac{\sqrt{n}\,\delta}{\sigma})}; k-m-t+1, m\right] d\Phi(y) . \tag{5.2}$$

From earlier work

$$\sup_{\Omega} E_{\underset{\sim}{\lambda}}[S(n)] = k \int_{-\infty}^{\infty} \{\Phi(y+d)\}^{k-1}\, I\left[\frac{\Phi(y)}{\Phi(y+d)}; k-m, m\right] d\Phi(y). \tag{5.3}$$

Setting the right-hand side of (5.3) equal to $1+\varepsilon$ and solving for d and then equating the right-hand side of (5.2) to P^* gives the proper sample size required to guarantee both (2.2) and the maximum expected number of selected populations is less than $1+\varepsilon$. Table I gives the values of $\frac{\sqrt{n}\,\delta}{\sigma}$ required to implement the rule for selected k, m and t.

II. Estimators, T_{in}, having noncentral chi square and noncentral F distributions arise in problems of selection from multivariate normal populations with known and unknown covariance matrices. See Gupta (1966), Alam and Rizvi (1966) and Gupta and Studden (1970). For an example of restricted subset selection from noncentral chi square populations with $t = 1$ see Santner (1973).

Suppose that T_{in} has a noncentral F cdf with parameters p and q and noncentrality parameter λ_i to be abbreviated $F_{p,q}(y|\lambda_i)$. Both p and q may depend on n but the dependence is suppressed for ease of notation. Here $\Lambda = [0,\infty)$, $G_n(y|\lambda) = F_{p,q}(y|\lambda)$ and $g_n(y|\lambda) = dF_{p,q}(y|\lambda)/dy = f_{p,q}(y|\lambda)$. Take

$$p(\lambda) = \begin{cases} \lambda - \delta_1, & 0 \le \lambda \le \delta_1\delta_2/(\delta_2 - 1), \quad (\delta_1 > 0, \ \delta_2 > 1) \\ \lambda/\delta_2, & \lambda > \delta_1\delta_2/(\delta_2 - 1) \end{cases}$$

and $h_n(x) = d^{1/n}x$, $d > 1$. Hence

Table I. Lists values of $\frac{\sqrt{n}\ \delta}{\sigma}$ required to attain P^* levels .75, .90 and .975 by rules (5.1) with $d = .4(.4)1.6$ for various k, m and t.

		d	.4			.8		
			P^*			P^*		
k	m	t	.75	.90	.975	.75	.90	.975
4	2	2	.419	1.121	1.914	.136	.818	1.585
6	2	2	.853	1.498	2.234	.638	1.261	1.960
	3	2	.797	1.457	2.203	.447	1.089	1.828
	4	2	.791	1.453	2.203	.398	1.054	1.804
8	2	2	1.080	1.701	2.406	.894	1.492	2.164
		3	.656	1.250	1.921	.458	1.025	1.664
	3	2	1.015	1.650	2.375	.698	1.308	2.007
		3	.596	1.203	1.898	.263	.848	1.523
	4	2	1.005	1.644	2.375	.627	1.257	1.976
10	2	2	1.230	1.837	2.531	1.061	1.644	2.304
		3	.835	1.410	2.062	.660	1.207	1.828
		4	.546	1.105	1.750	.361	.892	1.492
	3	2	1.162	1.781	2.492	.865	1.460	2.140
		3	.771	1.359	2.031	.462	1.023	1.671
		4	.484	1.058	1.718	.164	.712	1.343
	4	2	1.150	1.773	2.484	.785	1.396	2.093
		3	.759	1.351	2.031	.387	.966	1.632
15	2	2	1.468	2.054	2.726	1.323	1.886	2.523
		3	1.103	1.652	2.281	.954	1.478	2.070
		4	.853	1.380	1.984	.701	1.199	1.765
	3	2	1.395	1.994	2.679	1.130	1.701	2.351
		3	1.031	1.593	2.234	.759	1.293	1.898
		4	.783	1.324	1.945	.505	1.011	1.601
	4	2	1.377	1.982	2.671	1.041	1.625	2.296
		3	1.015	1.582	2.234	.669	1.218	1.843
		4	.767	1.314	1.945	.481	.943	1.554
20	2	2	1.619	2.195	2.851	1.486	2.039	2.664
		3	1.264	1.880	2.406	1.128	1.642	2.214
		4	1.029	1.539	2.125	.891	1.375	1.925
	3	2	1.543	2.128	2.796	1.298	1.853	2.492
		3	1.189	1.736	2.367	.937	1.457	2.039
		4	.953	1.476	2.085	.701	1.187	1.757
	4	2	1.521	2.115	2.789	1.204	1.771	2.421
		3	1.169	1.722	2.359	.842	1.375	1.984
		4	.934	1.466	2.078	.607	1.109	1.695

Table I (cont.)

		d	1.2			1.6		
			P*			P*		
k	m	t	.75	.90	.975	.75	.90	.975
4	2	2	*	.605	1.351	*	.476	1.218
6	2	2	.521	1.130	1.812	.464	1.070	1.750
	3	2	.181	.800	1.500	.009	.613	1.289
	4	2	.029	.675	1.414	*	.343	1.062
8	2	2	.802	1.390	2.046	.763	1.347	2.000
		3	.356	.910	1.531	.310	.859	1.476
	3	2	.490	1.076	1.742	.377	.949	1.593
		3	.031	.587	1.218	*	.435	1.039
	4	2	.316	.921	1.617	.104	.687	1.343
10	2	2	.984	1.558	2.203	.954	1.525	2.164
		3	.576	1.113	1.710	.543	1.074	1.664
		4	.270	.789	1.367	.231	.746	1.320
	3	2	.668	1.257	1.898	.601	1.160	1.781
		3	.271	.804	1.406	.173	.693	1.273
		4	*	.472	1.062	*	.343	.906
	4	2	.513	1.095	1.765	.351	.912	1.546
		3	.095	.648	1.281	*	.437	1.031
15	2	2	1.263	1.820	2.445	1.242	1.796	2.414
		3	.892	1.406	1.796	2.870	1.380	1.945
		4	.636	1.125	1.761	.612	1.097	1.640
	3	2	.994	1.541	2.156	.935	1.474	2.078
		3	.617	1.125	1.687	.556	1.052	1.601
		4	.357	.835	1.382	.292	.757	1.281
	4	2	.824	1.377	2.007	.716	1.252	1.859
		3	.444	.960	1.539	.332	.826	1.375
		4	.183	.671	1.242	.062	.525	1.054
20	2	2	1.435	1.980	2.593	1.418	1.962	2.570
		3	1.076	1.582	2.140	1.059	1.560	2.117
		4	.838	1.312	1.843	.820	1.291	1.820
	3	2	1.180	1.716	2.320	1.134	1.664	2.257
		3	.818	1.314	1.859	.771	1.257	1.789
		4	.579	1.039	1.566	.530	.980	1.492
	4	2	1.017	1.554	2.171	.933	1.457	2.054
		3	.650	1.152	1.710	.566	1.050	1.578
		4	.413	.875	1.418	.324	.765	1.277

This choice of d insures that the probability level P can even be attained over all Ω no matter what n is used.

$$\Omega^t(p) = \Omega_1 \cap \Omega_2 \quad \text{with} \quad \begin{cases} \Omega_1 = \{\underset{\sim}{\lambda} \mid \lambda_{[k-t+1]} - \lambda_{[k-t]} \geq \delta_1\} \\ \\ \Omega_2 = \{\underset{\sim}{\lambda} \mid \lambda_{[k-t+1]} \geq \delta_2 \lambda_{[k-t]}\} \end{cases}$$

$$\Lambda' = [\delta_1, \infty)$$

$R(n)$: Select $\pi_i \Longleftrightarrow T_{in} \geq \max\{T_{[k-m+1]}, d^{-1/n} T_{[k]}\}$ and

$$g_n(y|\lambda) = \begin{cases} \dfrac{e^{-\lambda/2}}{\Gamma(q/2)} \displaystyle\sum_{r=0}^{\infty} \dfrac{y^{\frac{p}{2}+r-1}\,\Gamma(\frac{p}{2}+\frac{q}{2}+1)\lambda^r}{(1+y)^{\frac{p}{2}+\frac{q}{2}+r}\,\Gamma(\frac{p}{2}+r)2^r r!}, & y > 0 \\ \\ 0, & y \leq 0 . \end{cases}$$

Since $\{F_{p,q}(y|\lambda) | \lambda \geq 0\}$ is a stochastically increasing family. Theorem 3.1 applies to give $\inf\limits_{\Omega^t(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)] = \inf\limits_{\lambda \geq \delta_1} \beta(\lambda,t,n)$ where $\beta(\lambda,t,n)$ is defined by

$$\begin{cases} \displaystyle\int_0^{\infty} \{F_{p,q}(yd^{1/n}|\lambda-\delta_1)\}^{k-t}\, I\left[\dfrac{F_{p,q}(y|\lambda-\delta_1)}{F_{p,q}(yd^{1/n}|\lambda-\delta_1)};\ k-m-t+1, m\right] dF_{p,q}(y|\lambda), & \lambda \in I_1 \\ \\ \displaystyle\int_0^{\infty} \{F_{p,q}(yd^{1/n}|\delta_2^{-1}\lambda)\}^{k-t}\, I\left[\dfrac{F_{p,q}(y|\delta_2^{-1}\lambda)}{F_{p,q}(yd^{1/n}|\delta_2^{-1}\lambda)};\ k-m-t+1, m\right] dF_{p,q}(y|\lambda), & \lambda \in I_2 \end{cases}$$

where

$$\begin{cases} I_1 = [\delta_1,\ \delta_1\delta_2/(\delta_2 - 1)] \\ \\ I_2 = [\delta_1\delta_2/(\delta_2 - 1),\ \infty) . \end{cases}$$

<u>Theorem 5.1</u>. For any $1 < d < \delta_2$ and $n \geq 1$, $\beta(\lambda,t,n)$ is nondecreasing in λ on I_1 and nonincreasing in λ on I_2. Hence

$$\inf_{\lambda \geq \delta_1} \beta(\lambda,t,n) = \beta\left(\frac{\delta_1\delta_2}{\delta_2 - 1}, t, n\right) .$$

<u>Proof</u>. A piecewise application of Theorem 3.2 will be made on I_1 and I_2. The following argument shows the method as used on I_1. Since $dF_{p,q}(y|\lambda)/d\lambda = \frac{-1}{q-2} f_{p+2,q-2}(y|\lambda)$ it can be seen that (3.5) reduces to

$$d^{1/n} f_{p+2,q-2}(y|\lambda) f_{p,q}(yd^{1/n}|\lambda-\delta_1) \leq f_{p+2,q-2}(y|\lambda-\delta_1)\cdot f_{p,q}(y|\lambda)$$

for all $y > 0$ and $\lambda \in I_1$

$$\Longleftrightarrow \quad \frac{1}{y}\,\frac{f_{p+2,q-2}(y|\lambda)}{f_{p,q}(y|\lambda)} \leq \frac{1}{d^{1/n}y}\,\frac{f_{p+2,q-2}(yd^{1/n}|\lambda-\delta_1)}{f_{p,q}(yd^{1/n}|\lambda-\delta_1)} \quad \text{for all } y > 0 \text{ and } \lambda \in I_1.$$

But

$$1 < d < \delta_2 \Rightarrow \frac{\delta_1\delta_2}{\delta_2 - 1} < \frac{\delta_1 d}{d-1}$$

$$\Rightarrow \lambda < \delta_1 d/(d-1) \quad \forall \quad \lambda \in I_1$$

$$\Rightarrow \lambda \leq \frac{\delta_1 d^{1/n}(1+y)}{d^{1/n} - 1} \quad \forall \quad y > 0 \quad \text{since} \quad \frac{d}{d-1} \leq \frac{d^{1/n}}{d^{1/n} - 1} \quad \forall \quad n \geq 1$$

$$\Rightarrow y(1 + d^{1/n}y) > (1+y)d^{1/n}(\lambda - \delta_1)$$

$$\Rightarrow \frac{\lambda y}{1+y} > \frac{d^{1/n}y(\lambda - \delta_1)}{(1+yd^{1/n})} \quad \forall \quad y \geq 0,\ \lambda \in I_1 \text{ and } n \geq 1 .$$

An application of problem 7.4 in Lehmann (1957) shows that $\frac{1}{y}\frac{f_{p+2,q-2}(y|\lambda)}{f_{p,q}(y|\lambda)}$ is $\downarrow$ in $\frac{\lambda y}{1+y}$ for fixed p and q and hence (3.5) holds. A similar argument gives (3.6) and hence completes the proof.

Finally since $\delta_1\delta_2/(\delta_2 - 1) \in \Lambda'$ we have for any $1 < d < \delta_2$, $\inf_{\Omega^t(p)} P[CS|R(n)] = \beta(\frac{\delta_1\delta_2}{\delta_2 - 1}, t,n) \to 1$ as $n \to \infty$. Expressions for $E_{\underset{\sim}{\lambda}}[S(n)]$ and $\sup_{\Omega} E_{\underset{\sim}{\lambda}}[S(n)$ are found in Santner (1973).

III. Suppose π_i has a uniform distribution on $[0,\lambda_i]$, $i = 1,\ldots,k$ then the problem of selecting one of the t populations with largest λ_i's is non-regular in the sense that $E_n^{\lambda_i} \neq E_n$ for all λ_i. Take $T_{in} = \max_{1\leq j\leq n} X_{ij}$, $p(\lambda) = \lambda\delta$ with $0 < \delta < 1$ and $h_n(x) = d^{1/n}$ where $d > 1$ and then $T_{in} \xrightarrow{p} \lambda_i$ as $n \to \infty$ and the problem can be formally stated in the language of Section 2 as

$$\Lambda = (0,\infty)$$

$$G_n(y|\lambda) = \begin{cases} 0, & y \leq 0 \\ (y/\lambda)^n, & 0 < y < \lambda_i \\ 1, & y \geq \lambda_i \end{cases}$$

$$\Omega^t(p) = \{\underset{\sim}{\lambda}|\lambda_{[k-t+1]} \geq \delta^{-1}\lambda_{[k-t]}\}$$

$$R(n): \text{ Select } \pi_i \Longleftrightarrow T_i \geq \max\{d^{-1/n}\, T_{[k]}, T_{[k-m+1]}\}\ .$$

Theorem 5.2

$$\inf_{\Omega^t(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)] = 1 + \delta^n\{\frac{1}{d^t k}(t-k)I(1/d;\ k-m-t+1,m) - \prod_{j=1}^{t}(\frac{k-m-t+j}{k-t+j})I(1-1/d;\ m,k-m+1)\} \tag{5.4}$$

Proof. Since $\{G_n(y|\lambda)|\lambda \in (0,\infty)\}$ is a stochastically increasing family for each n, Theorem 3.1 applies to give

$$\inf_{\Omega^t(p)} P_{\underset{\sim}{\lambda}}[CS|R(n)] = \inf_{\lambda>0} \beta(\lambda,t,n) \quad \text{where}$$

$$\beta(\lambda,t,n) = T_1(\lambda) + T_2(\lambda) + T_3(\lambda) \quad \text{and}$$

$$T_1(\lambda) = \sum_{j=k-t-m+1}^{k-t} \binom{k-t}{j} \int_0^{\lambda\delta/d^{1/n}} (\frac{y}{\lambda\delta})^{nj} \left\{d(\frac{y}{\lambda\delta})^n - (\frac{y}{\lambda\delta})^n\right\}^{k-t-j} d(\frac{y}{\lambda})^{nt}$$

$$T_2(\lambda) = \sum_{j=k-t-m+1}^{k-t} \binom{k-t}{j} \int_{\lambda\delta/d^{1/n}}^{\lambda\delta} (\frac{y}{\lambda\delta})^{nj} \left\{1 - (\frac{y}{\lambda\delta})^n\right\}^{k-t-j} d(\frac{y}{\lambda})^{nt}$$

$$T_3(\lambda) = \int_{\lambda\delta}^{\lambda} d(\frac{y}{\lambda})^{nt} .$$

A change of variables and a lengthy but routine calculation shows all $T_j(\lambda)$'s to be independent of λ and gives (5.4) completing the proof.

Either by direct examination of (5.4) or an appeal to Theorem 4.1 with $\lambda_0 = 1$ shows $\inf_{\Omega^t(p)} P[CS|R(n)] \to 1$ as $n \to \infty$. Hence any $(P^*, p(\lambda))$ requirements (2.2) can be achieved. All usual monotonicity properties hold for $R(n)$

and details of their derivation can be found in Santner (1973). In particular,

$$\sup_{\Omega} E_{\underset{\sim}{\lambda}}[S(n)] = k\{\frac{[1+kd-k]}{k} I(1/d;\ k-m,m) + \frac{m}{k} I(1 - 1/d;\ m+1,k-m)\} .$$

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